2665, 2670 [Jan., Feb., Sept., and Dec., 1918]. **Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.**

A telegraph wire which weighs 1/10 of a pound per yard is stretched between poles on a level ground so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

III. Solution by J. B. Reynolds, Lehigh University.

Choosing any small portion of the wire of length as resolving tangentially and normally to the curve of equilibrium and passing to differentials we get the fundamental equations

$$T\frac{d\varphi}{ds} = w \cos \varphi, \qquad \frac{dT}{ds} = w \sin \varphi \tag{1}$$

in which T is the tension in the wire, w the weight per foot, φ the angle the tangent makes with the horizontal and s the length of the curve measured from the lowest point.

Dividing and integrating, we find

$$T = T_0 \sec \varphi \tag{2}$$

T_0 being tension at lowest point. Putting this value of T in the second equation above we find

$$ws = T_0 \tan \varphi. \tag{3}$$

Now since $\tan \psi = dy/dx$ and $ds/dx = \sqrt{1 + (dy/dx)^2}$ we may, from (3), arrive at the equation

$$y = \frac{T_0}{2w}\{e^{wx/T_0} + e^{-wx/T_0}\} - \frac{T_0}{w}, \tag{4}$$

y being measured vertically from the lowest point of the curve, which is taken as the origin. Likewise, from these equations we may find

$$T_0 = \frac{w}{8d}(l^2 - 4d^2), \tag{5}$$

where l is the length of wire between poles and d is the greatest dip, and

$$l = h\left\{1 + \frac{8}{3}\left(\frac{d}{h}\right)^2 - \frac{224}{45}\left(\frac{d}{h}\right)^4 \cdots\right\}, \tag{6}$$

in which h is the horizontal distance between poles. For this problem, $T_0 = 140$, $w = 1/30$; so, by (4),

$$y = 2100\{e^{x/4200} + e^{-x/4200}\} - 4200.$$

Expanding to x^2, we find, when $y = 3$, $x^2 = 25200$; whence $x = 159$ ft., $2x = 318$ ft., as a good approximation of the distance between poles.

If we use equation (5), we have $l^2 = 100836$; whence $l = 317$ ft.—a nearer value. Finally by using (6) we get $h = 316 +$ ft.

2679 [Feb., 1918]. **Proposed by J. W. LASLEY, JR., University of North Carolina.**

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

Solution by C. E. Githens, Wheeling, W. Va.

Let CD be the perpendicular from the point C to any chord AB of the given circle, and let EC and CH be the perpendiculars from C to the tangents to the circle at the points A and B respectively. To prove $EC \times HC = CD^2$. The quadrilaterals $AECD$ and $BHCD$ are similar since the angles of the one are respectively equal to the angles of the other, that is, angle EAD = angle DBH, angle CDB = angle ADC = angle AEC = angle CHB, being right angles. Hence, the corresponding sides are proportional; that is, $EC : CD = CD : CH$, or $EC \times CH = CD^2$.

Also solved by C. A. Barnhart, Paul Capron, L. E. Mensenkamp, Roger A. Johnson, I. J. Fajans, Samuel Cohen, O. E. Simonsfn, I. Millenky,

J. V. Fazio, S. Weinberger, N. P. Pandya, H. L. Olson, Elijah Swift, W. L. Lord, J. W. Lasley, T. R. Thomson, Abraham Platman, H. Gladstone, Peter Pumo, Isadore Rose, R. M. Mathews, J. L. Riley.

2680 [March, 1918]. **Proposed by C. C. Yen, Tangshan, North China.**

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. (From Joseph Edward's *Elementary Treatise on Differential Calculus.*)

Solution by L. E. Mensenkamp, Freeport, Illinois.

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Now, it is easily shown that a quadrilateral inscribed in an ellipse is a parallelogram when, and only when, the opposite vertices are symmetrical with respect to the origin. In other words, the diagonals of an inscribed parallelogram pass through the center of the ellipse. Let us denote one vertex of the parallelogram (which for convenience we may assume to be in the first quadrant) by $P_1 = (x_1, y_1)$. Let an adjacent vertex be $P_2 = (x_2, y_2)$. Then, the vertex opposite P_1 will be $P_3 = (-x_1, -y_1)$, and that opposite P_2 will be $P_4 = (-x_2, -y_2)$.

We may call P_2P_3 the base of the parallelogram. Its equation may be written

$$(y_1 + y_2)x - (x_1 + x_2)y - (x_1 + x_2)y_1 + (y_1 + y_2)x_1 = 0.$$

The equation in this form enables us to apply the usual formula for the distance from a point to a line to obtain the distance from P_1 to the line P_2P_3, which is the altitude of the parallelogram. This expression for the altitude, after some reduction, becomes

$$D = \frac{2(x_1y_2 - x_2y_1)}{\sqrt{(y_1 + y_2)^2 + (x_1 + x_2)^2}}. \tag{1}$$

Multiplying D by the length of the base P_2P_3, we find the area of the parallelogram to be $A = 2(x_1y_2 - x_2y_1)$. Making use of the fact that these points lie on the ellipse, and assuming for the moment that P_2 lies above the X-axis, the area becomes

$$A = \frac{2b}{b}(x_1\sqrt{a^2 - x_2^2} - x_2\sqrt{a^2 - x_1^2}).$$

The condition for a maximum is that $\partial A/\partial x_1 = 0$, and $\partial A/\partial x_2 = 0$. Both of these conditions lead to the same equation, namely,

$$x_1x_2 = -\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}. \tag{2}$$

Therefore,

$$\frac{y_1y_2}{x_1x_2} = \frac{y_1y_2}{-\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}} = -\frac{b^2}{a^2}.$$

The last member of this equation follows from the elimination of y_1 and y_2 by means of the equation for the ellipse. Since y_1/x_1 and y_2/x_2 represent the slopes of the diagonals of the parallelogram, and since their product equals $-b^2/a^2$, it is seen (Bôcher, *Plane Analytic Geometry*, p. 153) that the diagonals are conjugate diameters of the ellipse.

If P_2 is assumed to be below the X-axis, we must remember to use

$$y_2 = -\frac{b}{a}\sqrt{a^2 - x_2^2},$$

but the final result is the same. It is evident that the equation (2) gives a maximum, and not a minimum, parallelogram; for, assuming P_2 temporarily fixed while P_1 varies, we see that the area is zero when P_1 coincides with either P_2 or P_4.

Also solved by J. B. Reynolds.

2681 [March, 1918]. **Proposed by Philip Franklin, College of the City of New York.**

Given n letters of one kind and $n - 1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?